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Abstract. A nonlinear superposition formula of the Novikov–Veselov equation is proved under certain conditions. Some particular solutions of the Novikov–Veselov equation are given as an illustrative application of the obtained result.

1. Introduction

There are two remarkable generalizations of the celebrated Korteweg–de Vries equation $u_t + 6uu_x + u_{xxx} = 0$ to versions in two spatial and one temporal (i.e. $2 + 1$) dimensions. One is the Kadomtsev–Petviashilli (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \tag{1}$$

where $\sigma^2 = \pm 1$. The other is [1, 2]

$$2u_t + u_{xxx} + u_{yyy} + 3(u\partial_y^{-1}u_x)_x + 3(u\partial_x^{-1}u_y)_y = 0\,. \tag{2}$$

We refer to (2) as the Novikov–Veselov (or Veselov–Novikov) equation.

As is known, many integrable nonlinear equations, such as the KdV in 1+1 and the KP in 2+1 dimensions share some common features, among which are the existences of hierarchies of equations solvable via the inverse scattering transform (IST), Lax representation, Backlund transformations etc. It has been shown that the NV equation shares some features as above [1–7]. For example, the NV equation is solvable via IST (see, e.g. [3]) and has a Backlund transformation and the soliton-like solutions [6, 7].

The purpose of this paper is to prove a nonlinear superposition formula of the NV equation under certain conditions. As we know, it is, in general, invalid for nonlinear differential equations to superpose their solutions although two solutions of a linear differential equation may be linearly superposed. However, for a nonlinear integrable equation, we can usually obtain a nonlinear superposition formula from the commutability of its BT, and the resulting nonlinear superposition formula enables one to get some exact solutions by the purely algebraic operations. Unfortunately, a rigorous proof of the commutability of the BT for a general nonlinear integrable equation is lacking [8, 9]. Therefore, it is neccessary to prove a nonlinear superposition formula directly.

This paper is organized as follows. In section 2, a nonlinear superposition formula of the NV equation is proved under certain conditions. Some particular solutions of the NV equation are given in section 3 as an application of the obtained result. In section 4, some concluding remarks are given. Finally, we list some bilinear operator identities in the appendix which are used in the paper.

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2. Nonlinear superposition formula of the NV equation

The NV equation (2) may be rewritten as the following generalized Hirota equation [4, 6]

$$D_x[(D_x^3D_y + 2D_tD_y)f \cdot f] \cdot f^2 + D_y(D_xD_y^3f \cdot f) \cdot f^2 = 0 \tag{3}$$

through the dependent variable transformation $u = 2(\ln f)_{xy}$, where the bilinear operator $D_x^l D_t^m D_y^n$ is defined as follows

$$D_x^l D_t^m D_y^n a(x,t,y) \cdot b(x,t,y) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n a(x,t,y)b(x',t',y')|_{x'=x,t'=t,y'=y}\,.$$

In [6], a bilinear BT with three parameters for (3) was presented

$$\begin{aligned} &(D_xD_y - \mu D_x - \lambda D_y + \lambda\mu)f \cdot f' = 0 \\ &(2D_t + D_x^3 + D_y^3 + 3\lambda^2 D_x - 3\lambda D_x^2 + 3\mu^2 D_y - 3\mu D_y^2 + \gamma)f \cdot f' = 0 \end{aligned} \tag{4}$$

where λ, μ and γ are constants. In what follows, we set $\mu = \gamma = 0$ in (4) for the sake of convenience in calculation. In this case, (4) becomes

$$(D_xD_y - \lambda D_y)f \cdot f' = 0 \qquad (2D_t + D_x^3 + D_y^3 + 3\lambda^2 D_x - 3\lambda D_x^2)f \cdot f' = 0\,. \tag{4'}$$

In this section, under certain conditions, we shall establish a nonlinear superposition formula of (3). To this end, let f_0 be a solution of (3), $f_0 \neq 0$. Suppose that $f_i (i = 1, 2)$ is a solution of (3) which is related by f_0 under BT (4′) with λ_i, i.e. $f_0 \xrightarrow{\lambda_i} f_i (i = 1, 2)$, and that f_{12} is defined by

$$D_y f_0 \cdot f_{12} = kD_y f_1 \cdot f_2 \qquad \text{where } k \text{ is a non-zero constant}\,. \tag{5}$$

From these assumptions and similar to the deduction of [10], we have, by use of (A1) and (5)

$$\begin{aligned} 0 &= [(D_xD_y - \lambda_1 D_y)f_0 \cdot f_1]f_2 - [(D_xD_y - \lambda_2 D_y)f_0 \cdot f_2]f_1 \\ &= -f_{0y}[D_x f_1 \cdot f_2 + (\lambda_1 - \lambda_2)f_1f_2 + \frac{1}{k}D_x f_0 \cdot f_{12} - \frac{1}{k}(\lambda_1 + \lambda_2)f_0f_{12}] \\ &\quad + \frac{1}{2}f_0[D_x f_1 \cdot f_2 + (\lambda_1 - \lambda_2)f_1f_2 + \frac{1}{k}D_x f_0 \cdot f_{12} - \frac{1}{k}(\lambda_1 + \lambda_2)f_0f_{12}]_y \end{aligned}$$

from which it follows that

$$D_x f_1 \cdot f_2 + (\lambda_1 - \lambda_2)f_1f_2 + \frac{1}{k}D_x f_0 \cdot f_{12} - \frac{1}{k}(\lambda_1 + \lambda_2)f_0f_{12} = c_1(t,x)f_0^2 \tag{6}$$

where $c_1(t,x)$ is some function of t, x. Here and in the following, we assume that there exists a f_{12} determined by (5) such that $c_1(t,x) = 0$, i.e.

$$D_x f_1 \cdot f_2 + (\lambda_1 - \lambda_2)f_1f_2 + \frac{1}{k}D_x f_0 \cdot f_{12} - \frac{1}{k}(\lambda_1 + \lambda_2)f_0f_{12} = 0\,. \tag{6'}$$

In this case, similar to the deduction of [10], we have

$$(D_xD_y - \lambda_2 D_y)f_1 \cdot f_{12} = 0 \qquad (D_xD_y - \lambda_1 D_y)f_2 \cdot f_{12} = 0\,.$$

Next, from

$$[(2D_t+D_x^3+D_y^3+3\lambda_1^2D_x-3\lambda_1D_x^2)f_0\cdot f_1]f_2-[(2D_t+D_x^3+D_y^3+3\lambda_2^2D_x-3\lambda_2D_x^2)f_0\cdot f_2]f_1 = 0$$

we have, by using (A2), (A3), (5) and (6′)

$$-\left[2D_t+\frac{1}{4}D_x^3+\frac{1}{4}D_y^3+\frac{3}{2}(\lambda_1^2+\lambda_2^2)D_x+\frac{3}{4}(\lambda_1-\lambda_2)D_x^2\right]f_1\cdot f_2 \\ +\frac{1}{k}\left[\frac{3}{4}D_x^3-\frac{3}{4}D_y^3-\frac{9}{4}(\lambda_1+\lambda_2)D_x^2+\frac{3}{2}(\lambda_1+\lambda_2)^2D_x\right]f_0\cdot f_{12}=0\,. \qquad (7)$$

Moreover, from

$$[(D_xD_y-\lambda_1D_y)f_0\cdot f_1]_xf_2-[(D_xD_y-\lambda_2D_y)f_0\cdot f_2]_xf_1 \\ +\tfrac{1}{2}(\lambda_1-\lambda_2)\{[(D_xD_y-\lambda_1D_y)f_0\cdot f_1]f_2+[(D_xD_y-\lambda_2D_y)f_0\cdot f_2]f_1\}=0$$

we get, by using (A4), (5) and (6′), that

$$\frac{1}{k}\left[-\frac{1}{4}D_x^2D_y+\frac{1}{2}(\lambda_1+\lambda_2)D_xD_y-\frac{1}{4}(\lambda_1+\lambda_2)^2D_y\right]f_0\cdot f_{12} \\ +\left[\frac{1}{4}D_x^2D_y+\frac{1}{2}(\lambda_1-\lambda_2)D_xD_y+\frac{1}{4}(\lambda_1-\lambda_2)^2D_y\right]f_1\cdot f_2=0\,. \qquad (8)$$

Furthermore, from

$$[(2D_t+D_x^3+D_y^3+3\lambda_1^2D_x-3\lambda_1D_x^2)f_0\cdot f_1]_yf_2-[(2D_t+D_x^3+D_y^3+3\lambda_2^2D_x-3\lambda_2D_x^2)f_0\cdot f_2]_yf_1 \\ +3[(D_xD_y-\lambda_1D_y)f_0\cdot f_1]_{xx}f_2-3[(D_xD_y-\lambda_2D_y)f_0\cdot f_2]_{xx}f_1 \\ +3(\lambda_1-\lambda_2)\{[(D_xD_y-\lambda_1D_y)f_0\cdot f_1]_xf_2+[(D_xD_y-\lambda_2D_y)f_0\cdot f_2]_xf_1\}=0$$

we can deduce, by use of (A5), (A6), (5), (6′) and (8) and by a tedious calculation,

$$0=f_{0y}\left\{\frac{1}{k}\left[2D_t-\frac{1}{2}D_y^3+D_x^3-3(\lambda_1+\lambda_2)D_x^2+3(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)D_x\right]f_0\cdot f_{12}\right. \\ \left.+\left[-2D_t+\frac{1}{2}D_y^3-D_x^3-3(\lambda_1-\lambda_2)D_x^2-3(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)D_x\right]f_1\cdot f_2\right\} \\ +f_0\left\{\frac{1}{k}\left[-D_t-\frac{1}{2}D_y^3+\frac{1}{4}D_x^3-\frac{3}{4}(\lambda_1+\lambda_2)D_x^2+\frac{3}{2}\lambda_1\lambda_2D_x\right]f_0\cdot f_{12}\right. \\ \left.+\left[-D_t-\frac{1}{2}D_y^3+\frac{1}{4}D_x^3+\frac{3}{4}(\lambda_1-\lambda_2)D_x^2-\frac{3}{2}\lambda_1\lambda_2D_x\right]f_1\cdot f_2\right\}_y$$

$$\overset{(7)}{=} f_{0y}\left\{\frac{1}{k}\left[2D_t+\frac{1}{4}D_y^3+\frac{1}{4}D_x^3-\frac{3}{4}(\lambda_1+\lambda_2)D_x^2+\frac{3}{2}(\lambda_1^2+\lambda_2^2)D_x\right]f_0\cdot f_{12}\right. \\ \left.+\left[\frac{3}{4}D_y^3-\frac{3}{4}D_x^3-\frac{9}{4}(\lambda_1-\lambda_2)D_x^2-\frac{3}{2}(\lambda_1-\lambda_2)^2D_x\right]f_1\cdot f_2\right\} \\ +f_0\left\{\frac{1}{k}\left[-D_t-\frac{1}{8}D_y^3-\frac{1}{8}D_x^3+\frac{3}{8}(\lambda_1+\lambda_2)D_x^2-\frac{3}{4}(\lambda_1^2+\lambda_2^2)D_x\right]f_0\cdot f_{12}\right. \\ \left.+\left[-\frac{3}{8}D_y^3+\frac{3}{8}D_x^3+\frac{9}{8}(\lambda_1-\lambda_2)D_x^2+\frac{3}{4}(\lambda_1-\lambda_2)^2D_x\right]f_1\cdot f_2\right\}_y$$

which implies that

$$\begin{aligned}&\left[2D_t+\tfrac{1}{4}D_y^3+\tfrac{1}{4}D_x^3-\tfrac{3}{4}(\lambda_1+\lambda_2)D_x^2+\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)D_x\right]f_0\cdot f_{12}\\&\qquad+k\left[\tfrac{3}{4}D_y^3-\tfrac{3}{4}D_x^3-\tfrac{9}{4}(\lambda_1-\lambda_2)D_x^2-\tfrac{3}{2}(\lambda_1-\lambda_2)^2D_x\right]f_1\cdot f_2=c_2(t,x)f_0^2\end{aligned}\tag{9}$$

where $c_2(t,x)$ is some function of t, x. Furthermore we assume that f_{12} determined by (5) is chosen such that $c_2(t,x)=0$. In this case, we have

$$\begin{aligned}&\left[2D_t+\tfrac{1}{4}D_y^3+\tfrac{1}{4}D_x^3-\tfrac{3}{4}(\lambda_1+\lambda_2)D_x^2+\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)D_x\right]f_0\cdot f_{12}\\&\qquad+k\left[\tfrac{3}{4}D_y^3-\tfrac{3}{4}D_x^3-\tfrac{9}{4}(\lambda_1-\lambda_2)D_x^2-\tfrac{3}{2}(\lambda_1-\lambda_2)^2D_x\right]f_1\cdot f_2=0\,.\end{aligned}\tag{9'}$$

Finally, we have

$$\begin{aligned}
&-[(2D_t+D_x^3+D_y^3+3\lambda_2^2D_x-3\lambda_2D_x^2)f_1\cdot f_{12}]f_0\\
&\quad=[(2D_t+D_x^3+D_y^3+3\lambda_1^2D_x+3\lambda_1D_x^2)f_1\cdot f_0]f_{12}\\
&\qquad-[(2D_t+D_x^3+D_y^3+3\lambda_2^2D_x-3\lambda_2D_x^2)f_1\cdot f_{12}]f_0\\
&\overset{(A2,A3)}{=}-2f_1D_tf_0\cdot f_{12}-3f_{1yy}D_yf_0\cdot f_{12}+3f_{1y}(D_yf_0\cdot f_{12})_y\\
&\qquad-\tfrac{1}{4}f_1[D_y^3f_0\cdot f_{12}+3(D_yf_0\cdot f_{12})_{yy}]-3f_{1xx}D_xf_0\cdot f_{12}\\
&\qquad+3f_{1x}(D_xf_0\cdot f_{12})_x-\tfrac{1}{4}f_1[D_x^3f_0\cdot f_{12}+3(D_xf_0\cdot f_{12})_{xx}]\\
&\qquad+3(\lambda_1^2-\lambda_2^2)f_{1x}f_0f_{12}-\tfrac{3}{2}(\lambda_1^2-\lambda_2^2)f_1(f_0f_{12})_x-\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)f_1D_xf_0\cdot f_{12}\\
&\qquad+3(\lambda_1+\lambda_2)f_{1xx}f_0f_{12}-3(\lambda_1+\lambda_2)f_{1x}(f_0f_{12})_x-3(\lambda_1-\lambda_2)f_{1x}D_xf_0\cdot f_{12}\\
&\qquad+3f_1(\lambda_1f_{0xx}f_{12}+\lambda_2f_0f_{12xx})\\
&\overset{(5,6')}{=}-2f_1D_tf_0\cdot f_{12}-3kf_{1yy}D_yf_1\cdot f_2+3kf_{1y}(D_yf_1\cdot f_2)_y\\
&\qquad-\tfrac{1}{4}f_1[D_y^3f_0\cdot f_{12}+3(D_yf_0\cdot f_{12})_{yy}]+3kf_{1xx}[D_x+(\lambda_1-\lambda_2)]f_1\cdot f_2\\
&\qquad-3kf_{1x}\{[D_x+(\lambda_1-\lambda_2)]f_1\cdot f_2\}_x-\tfrac{1}{4}f_1[D_x^3f_0\cdot f_{12}+3(D_xf_0\cdot f_{12})_{xx}]\\
&\qquad+3k(\lambda_1-\lambda_2)f_{1x}[D_x+(\lambda_1-\lambda_2)]f_1\cdot f_2-\tfrac{3}{2}(\lambda_1^2-\lambda_2^2)f_1(f_0f_{12})_x\\
&\qquad-\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)f_1D_xf_0\cdot f_{12}+3f_1(\lambda_1f_{0xx}f_{12}+\lambda_2f_0f_{12xx})\\
&\overset{(5)}{=}f_1\{-2D_tf_0\cdot f_{12}+3kf_{1yy}f_{2y}-3kf_{1y}f_{2yy}-\tfrac{1}{4}D_y^3f_0\cdot f_{12}-\tfrac{3}{4}k(D_yf_1\cdot f_2)_{yy}\\
&\qquad-3kf_{1xx}f_{2x}+3k(\lambda_1-\lambda_2)f_{1xx}f_2+3kf_{1x}f_{2xx}-6k(\lambda_1-\lambda_2)f_{1x}f_{2x}\\
&\qquad-\tfrac{1}{4}D_x^3f_0\cdot f_{12}-\tfrac{3}{4}(D_xf_0\cdot f_{12})_{xx}+3(\lambda_1-\lambda_2)^2f_{1x}f_2-\tfrac{3}{2}(\lambda_1^2-\lambda_2^2)(f_0f_{12})_x\\
&\qquad-\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)D_xf_0\cdot f_{12}+\tfrac{3}{2}(\lambda_1-\lambda_2)(D_xf_0\cdot f_{12})_x\\
&\qquad+\tfrac{3}{4}(\lambda_1+\lambda_2)[D_x^2f_0\cdot f_{12}+(f_0f_{12})_{xx}]\}\\
&\overset{(6')}{=}f_1\{-2D_tf_0\cdot f_{12}-\tfrac{1}{4}D_y^3f_0\cdot f_{12}-\tfrac{1}{4}D_x^3f_0\cdot f_{12}-\tfrac{3}{2}(\lambda_1^2+\lambda_2^2)D_xf_0\cdot f_{12}\\
&\qquad+\tfrac{3}{4}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}+3kf_{1yy}f_{2y}-3kf_{1y}f_{2yy}-\tfrac{3}{4}k(D_yf_1\cdot f_2)_{yy}
\end{aligned}$$

$$
\begin{aligned}
&-3f_{1xx}f_{2x}+3k(\lambda_1-\lambda_2)f_{1xx}f_2+3kf_{1x}f_{2xx}-6k(\lambda_1-\lambda_2)f_{1x}f_{2x}\\
&+3(\lambda_1-\lambda_2)^2 f_{1x}f_2+\tfrac34 k[(D_x+(\lambda_1-\lambda_2))f_1\cdot f_2]_{xx}\\
&-\tfrac32 k(\lambda_1-\lambda_2)[(D_x+(\lambda_1-\lambda_2))f_1\cdot f_2]_x\}\\
=\ & f_1\big[-2D_t f_0\cdot f_{12}-\tfrac14 D_y^3 f_0\cdot f_{12}-\tfrac14 D_x^3 f_0\cdot f_{12}-\tfrac32(\lambda_1^2+\lambda_2^2)D_x f_0\cdot f_{12}\\
&+\tfrac34(\lambda_1+\lambda_2)D_x^2 f_0\cdot f_{12}-\tfrac34 kD_y f_1\cdot f_2+\tfrac34 kD_x^3 f_1\cdot f_2+\tfrac94 k(\lambda_1-\lambda_2)D_x^2 f_1\cdot f_2\\
&+\tfrac32 k(\lambda_1-\lambda_2)^2 D_x f_1\cdot f_2\big]\overset{(9')}{=}0
\end{aligned}
$$

which implies that

$$(2D_t+D_x^3+D_y^3+3\lambda_2^2D_x-3\lambda_2D_x^2)f_1\cdot f_{12}=0\,.$$

Similarly, we can show that

$$(2D_t+D_x^3+D_y^3+3\lambda_1^2D_x-3\lambda_1D_x^2)f_2\cdot f_{12}=0\,.$$

Thus we have shown the nonlinear superposition formula (5) of the NV equation (3) under the conditions $c_1(t,x)=c_2(t,x)=0$, and f_{12} is a new solution of (3).

3. Some particular solutions of the NV equation

In this section, we are going to derive some exact solutions of the NV equation from the BT (4') and the nonlinear superposition formula (5). We seek particular solutions of the NV equation via the following steps. First, choose a given solution f_0 of (3). Secondly, from BT(4') we find out f_1 and f_2 such that $f_0\xrightarrow{\lambda_i}f_i(i=1,2)$ and further, get a particular solution $\widetilde{f_{12}}$ from (5). Then a general solution of (5) is $f_{12}=c(t,x)f_0+\widetilde{f_{12}}$ (where $c(t,x)$ is an arbitrary function of t,x). Finally we substitute f_{12} into (6) and (9). If $c(t,x)$ can be determined such that $c_1(t,x)=c_2(t,x)=0$, the corresponding f_{12} is a new solution of the NV equation (3). In what follows, we give four illustrative examples.

(*a*) It is easily verified that

$$
\begin{array}{ccccc}
 & \overset{p_1}{\nearrow} & 1+e^{\eta_1} & \overset{p_2}{\searrow} & \\
1 & & & & \dfrac{p_1-p_2}{p_1+p_2}-e^{\eta_1}+e^{\eta_2}-\dfrac{q_1-q_2}{q_1+q_2}e^{\eta_1+\eta_2}\,.\\
 & \underset{p_2}{\searrow} & 1+e^{\eta_2} & \underset{p_1}{\nearrow} &
\end{array}
$$

Therefore $(p_1-p_2)/(p_1+p_2)-e^{\eta_1}+e^{\eta_2}-(q_1-q_2)/(q_1+q_2)e^{\eta_1+\eta_2}$ is a solution of the NV equation (3), where $\eta_i=p_ix+q_iy-\frac12(p_i^3+q_i^3)t+\eta_i^0$ and p_i,q_i,η_i^0 are constants $(i=1,2)$.

(*b*) It is easily verified that

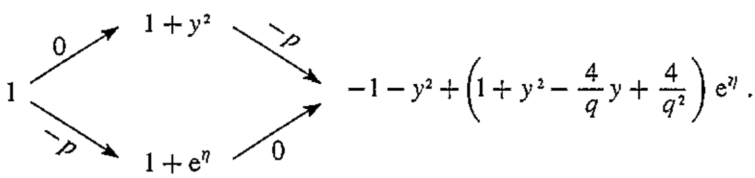
$$
\begin{array}{ccccc}
 & \overset{p_3}{\nearrow} & \dfrac{p_1-p_3}{p_1+p_3}-e^{\eta_1}+e^{\eta_3}-\dfrac{q_1-q_3}{q_1+q_3}e^{\eta_1+\eta_3} & \overset{p_2}{\searrow} & \\
1+e^{\eta_1} & & & & F\\
 & \underset{p_2}{\searrow} & \dfrac{p_1-p_2}{p_1+p_2}-e^{\eta_1}+e^{\eta_2}-\dfrac{q_1-q_2}{q_1+q_2}e^{\eta_1+\eta_2} & \underset{p_3}{\nearrow} &
\end{array}
$$

where

$$F = \frac{p_1 - p_3}{p_1 + p_3} - \frac{p_1 - p_2}{p_1 + p_2} - \frac{p_2 - p_3}{p_2 + p_3} - \frac{p_2 - p_3}{p_2 + p_3} e^{\eta_1} + \frac{p_1 - p_3}{p_1 + p_3} e^{\eta_2} - \frac{p_1 - p_2}{p_1 + p_2} e^{\eta_3} + \frac{q_1 - q_2}{q_1 + q_2} e^{\eta_1 + \eta_2} + \frac{q_2 - q_3}{q_2 + q_3} e^{\eta_2 + \eta_3} + \frac{q_3 - q_1}{q_1 + q_3} e^{\eta_1 + \eta_3} + \frac{(q_1 - q_3)(q_1 - q_2)(q_2 - q_3)}{(q_1 + q_3)(q_1 + q_2)(q_2 + q_3)} e^{\eta_1 + \eta_2 + \eta_3}$$

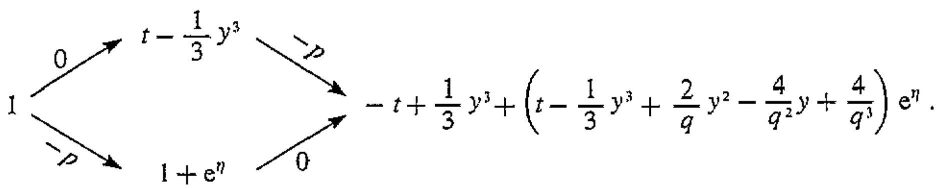
and $\eta_i = p_i x + q_i y - \frac{1}{2}(p_i^3 + q_i^3)t + \eta_i^0$, p_i, q_i, η_i^0 are constants ($i = 1, 2, 3$). Therefore F is a 3-soliton solution of (3).

(*c*) It is easily verified that



Therefore $-1 - y^2 + (1 + y^2 - (4/q)y + (4/q^2))e^{\eta}$ is a solution of the NV equation (3), where $\eta = px + qy - \frac{1}{2}(p^3 + q^3)t + \eta^0$ and p, q, η^0 are constants.

(*d*) It is easily verified that



Therefore $-t + \frac{1}{3}y^3 + (t - \frac{1}{3}y^3 + (2/q)y^2 - (4y/q^2) + (4/q^3))e^{\eta}$ is a solution of the NV equation (3), where $\eta = px + qy - \frac{1}{2}(p^3 + q^3)t + \eta^0$ and p, q, η^0 are constants.

4. Concluding remarks

We have shown a nonlinear superposition formula of the NV equation under certain conditions. As an application of the obtained result, we derived some particular solutions of the NV equation. To our knowledge, the solutions given in examples (*c*) and (*d*) are new, while the solutions given in examples (*a*) and (*b*) are two-soliton and three-soliton solutions respectively. Further the iterative use of superposition formula (5) enables one to obtain more solutions of the NV equation. It is noted that the NV equation is symmetric in x, y. Thus more solutions of the NV equation may also be obtained. For example, from the solutions derived in section 3, we know $-1 - x^2 + (1 + x^2 - (4/p)x + (4/p^2)) \exp(px + qy - \frac{1}{2}(p^3 + q^3)t + \eta^0)$ is also a solution of the NV equation (3). Finally, concerning (2), we can also consider the following dependent variable transformation

$$u = u_0 + 2(\ln f)_{xy} \qquad u_0 \text{ is a constant.}$$

In this case, (2) may be rewritten as

$$D_x[(D_x^3D_y + 2D_tD_y + 3u_0D_x^2)f \cdot f] \cdot f^2 + D_y[(D_xD_y^3 + 3u_0D_y^2)f \cdot f] \cdot f^2 = 0\,. \qquad (10)$$

Equation (10) is, in fact, a special form of the following equation:

$$D_x[(D_x^3D_y + \alpha_1 D_y^2 + \beta_1 D_xD_y + \delta_1 D_tD_y + \delta_2 D_x^2)f \cdot f] \cdot f^2 + D_y[(\alpha_2 D_xD_y^3 + \alpha_2\delta_2 D_y^2)f \cdot f] \cdot f^2 = 0 \qquad (11)$$

which is considered in [6]. Here $\alpha_1, \alpha_2, \beta_1, \delta_1, \delta_2$ are arbitrary constants. A BT for (11) is presented as follows [6]:

$$(D_xD_y - \mu D_x - \lambda D_y + \lambda\mu + \tfrac{1}{3}\delta_2)f \cdot f' = 0$$

$$(\delta_1 D_t + D_x^3 + \alpha_2 D_y^3 + \beta_1 D_x + \alpha_1 D_y + 3\lambda^2 D_x - 3\lambda D_x^2 + 3\mu^2 D_y - 3\mu D_y^2 + \gamma)f \cdot f' = 0$$

where λ, μ, γ are arbitrary constants. Similarly, we can also consider the corresponding nonlinear superposition formula.

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Appendix

The following bilinear operator identities hold for any arbitrary functions a, b, c and d:

$$(D_xD_ya \cdot b)c - (D_xD_ya \cdot c)b = -a_xD_yb \cdot c - a_yD_xb \cdot c + \tfrac{1}{2}a[(D_xb \cdot c)_y + (D_yb \cdot c)_x] \qquad (\text{A1})$$

$$(D_ta \cdot b)c - (D_ta \cdot c)b = -aD_tb \cdot c \qquad (\text{A2})$$

$$(D_z^3a \cdot b)c - (D_z^3a \cdot c)b = -3a_{zz}D_zb \cdot c + 3a_z(D_zb \cdot c)_z - \tfrac{1}{4}a[D_z^3b \cdot c + 3(D_zb \cdot c)_{zz}] \qquad (\text{A3})$$

$$(D_xD_ya \cdot b)_xc - (D_xD_ya \cdot c)_xb = -a_{xx}D_yb \cdot c - a_y(D_xb \cdot c)_x + \tfrac{1}{4}a[D_x^2D_yb \cdot c + (D_yb \cdot c)_{xx} + 2(D_xb \cdot c)_{xy}] \qquad (\text{A4})$$

$$(D_ta \cdot b)_yc - (D_ta \cdot c)_yb = a_tD_yb \cdot c - a_yD_tb \cdot c - \tfrac{1}{2}a[(D_yb \cdot c)_t + (D_tb \cdot c)_y] \qquad (\text{A5})$$

$$\begin{aligned}(D_x^3a \cdot b)_yc &- (D_x^3a \cdot c)_yb + 3(D_xD_ya \cdot b)_{xx}c - 3(D_xD_ya \cdot c)_{xx}b \\ = &-2a_{xxx}D_yb \cdot c - 3a_{xx}[(D_xb \cdot c)_y + (D_yb \cdot c)_x] + \tfrac{3}{2}a_x[D_x^2D_yb \cdot c \\ &+ 2(D_xb \cdot c)_{xy} + (D_yb \cdot c)_{xx}] - a_y[D_x^3b \cdot c + 3(D_xb \cdot c)_{xx}] + \tfrac{1}{4}a[(D_x^3b \cdot c)_y \\ &+ 3(D_x^2D_yb \cdot c)_x + 3(D_xb \cdot c)_{xxy} + (D_yb \cdot c)_{xxx}]\end{aligned} \qquad (\text{A6})$$

$$(D_y^3a \cdot b)_yc - (D_y^3a \cdot c)_yb = -2a_{yyy}D_yb \cdot c + \tfrac{1}{2}a_y[D_y^3b \cdot c + 3(D_yb \cdot c)_{yy}] - \tfrac{1}{2}a[D_y^3b \cdot c + (D_yb \cdot c)_{yy}]_y \qquad (\text{A7})$$

$$(D_xD_ya \cdot b)_xc + (D_xD_ya \cdot c)_xb = 2a_{xxy}bc - a_{xx}(bc)_y - \tfrac{1}{2}a_y[D_x^2b \cdot c + (bc)_{xx}] + \tfrac{1}{4}a[(bc)_{xxy} + (D_x^2b \cdot c)_y + 2(D_xD_yb \cdot c)_x]\,. \qquad (\text{A8})$$

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